'Generalisation is the heartbeat of mathematics, and appears in many forms. If teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalisations, then mathematical thinking is not taking place.'

(Mason, 1996, p.1)

Generalising

Why is generalising an important notion?

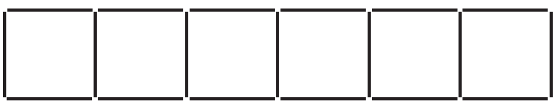
Identifying pattern and structure is at the heart of mathematics. Our role as educators is to choose materials and pedagogical approaches that enable learners to explore patterns in all areas of mathematics, in particular the *number and algebra* strand. As well as developing key understandings about the structure of mathematics, this approach also allows learners to discover the beauty of the discipline and to act as mathematicians (ACARA, 2017). Understanding the underlying structures of mathematics enables us to use its patterns to 'explain' and predict natural phenomena (Australian Education Council, 1991).

This paper provides guidelines on generalising and how leaders can support educators to develop an understanding of generalising, both for themselves and for their learners.

What is generalising?

When learners are faced with a mathematical problem they are often required to identify a pattern and then give a general rule for what they see.

For example, a linear pattern of 6 squares can be made using 19 matchsticks.



How many matchsticks would you need to make a pattern of 20, 100 or N squares long?
(Gilderdale & Piggott, 2011, used with permission, copyright University of Cambridge)

Generalising begins when learners start to notice patterns and spot aspects that stay the same when other aspects change. This shifts the focus in classroom mathematics from finding one correct numerical answer to recognising and constructing patterns and generalising mathematical relationships (important for algebraic work).

Learners also make *conjectures*: mathematical ideas that seem reasonable, but have not yet been proven.

For example, a student might notice 3 extra matchsticks are needed to add to the pattern of 6 squares to make the seventh square. Then 3 more are needed to make each subsequent square: a pattern of adding 3.

It is not enough though to just identify a pattern: learners need to be able to explain why the pattern works. This then leads to learners being able to generalise: being able to explain and apply the general rule in other instances.

For example, the student may note it takes 4 matchsticks to make one square and 3 more matchsticks to make 2 squares. So for 20 squares you would need 4 matchsticks for the first square and then 19 sets of 3 matchsticks to make the remaining squares (4 + 19 x 3 = 61). For any number of squares, N, you would need 4 matchsticks for the first square and then N minus 1 sets of 3 matchsticks for the remaining squares (4 + 3(N-1)) to obtain the total number of matchsticks.

But there are other ways to do it.

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Another way of seeing this pattern is to start with one matchstick. Then keep adding three matchsticks until you get the required number of squares. Hence, 1 + 3N is another generalisation for the problem of the squares and matchsticks.

Both expressions, 1 + 3N and 4 + 3 (N-1), describe the generalisation and provide students with an opportunity to use number properties in order to identify equivalent algebraic expressions. Ultimately, generalising involves engaging with, interpreting, working with and constructing algebraic text.

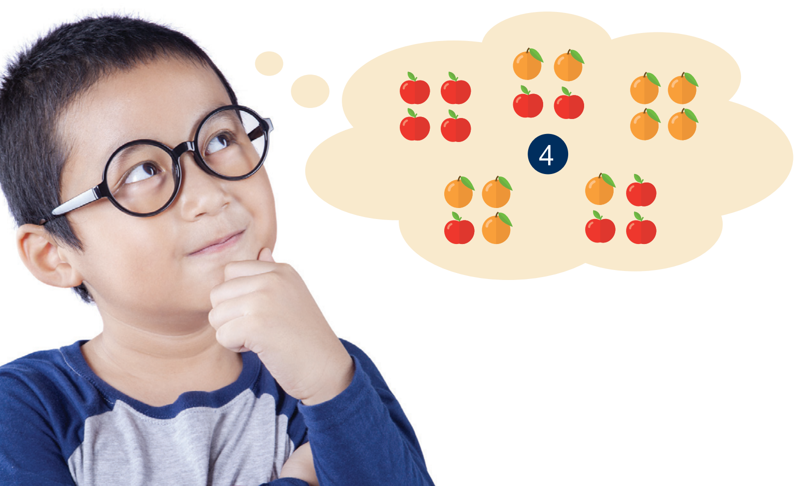
How can educators help learners to generalise?

Creating generalisations from number and arithmetic begins in the early years and progresses as learners develop their understanding of number sense, including basic number facts and meanings of the operations. Young learners learn how to put together numbers and take them apart again when they are given situations or 'problems' that have more than one possible solution.

For example, an educator could state that eight children were playing in the yard, either in the sand pit or at the water-play. The educator could share thinking aloud by proposing: *'There are 8 children at the sandpit but none at the water-play …Oh, wait, now there are 4 children at each. I wonder what other possible ways the children could be playing in those two areas?'*

By exploring complex problems and finding multiple possible solutions, learners are encouraged to use reasoning about what they have learnt for one situation and transfer it to another in order to find a solution that will *always* work—the essence of a generalisation.

Number sense is strengthened through discussions around questions such as:
- **What do you notice?**
- **Could this always be true?**
- **How could we represent that?**
- **Can you see/describe the pattern?**



Learners initially use diagrams, concrete materials and words to justify their own generalisations. As the explanations for their generalisations become more complicated, learners need to use other ways to represent them: this progresses into conventional algebraic notation. The reasoning that underlies the generalisation is an important bridge to algebraic thinking: progressing to formal notation too soon can result in later difficulties and misunderstandings.

Our role as educators is to help learners to make conjectures by articulating, editing and refining their mathematical ideas. True/false sentences support these explorations (for example see Victorian Department of Education and Training, 2015).

How can leaders support their staff?

Generalising can be promoted by creating collaborative structures and opportunities through which educators explore a range of contexts and multiple representations for sophisticated problems with more than one possible solution. Leaders support enhanced practice when they provide opportunities for educators to use their professional learning and research to design and trial high quality models, focussing on questions that foster generalising, for example 'What do you notice?' 'Could this always be true?' 'How would you represent that?'

It is important for educators to build pedagogical content knowledge around the developmental ideas for generalising and to recognise and address learners' difficulties before moving into symbolic representation.

Leaders should consider how they work with their staff to incorporate the big ideas in number into common agreements around planning, teaching and assessment at their site.

Hence, leaders could support their staff by:

- providing professional learning focused on mathematical problem solving
- setting up shared planning processes based on the big ideas in number
- observing how teachers model generalising in their classrooms
- encouraging teachers to showcase intentional teaching of generalising (eg through reflective questions)
- using audit tools and student surveys to develop a whole school picture about generalising in mathematics.

Look for learning that involves:

Teachers as powerful role models

When teachers enthusiastically share and talk about mathematics, they inspire learners to think mathematically. When observing lessons, look for learners who:

- 'have a go' at problem solving
- make generalisations and test them
- can attempt to justify their generalisations
- are willing to learn from their mistakes
- make connections to their lives in local, global and broader contexts.

Differentiation

The teacher's role is to respond to each individual student's learning needs and to direct the next step. What opportunities do teachers provide for individual or small groups of students to use numeracy and mathematical understandings to gather evidence, make conjectures and generalise?

Teaching intentionally

- Are teachers using test items (eg NAPLAN, PAT-M) to give students more experiences of how generalising is a valued skill in mathematics?
- Are teachers using the generalising common misunderstanding tools? (Siemon, 2009)
- Are students becoming more aware of what they are learning, how well they are going, and what they need to do next to be successful? (Hattie & Timperley, 2007)

What rule works for calculating the number of sticks required to make any of the images in this growing pattern?

- ○ 5 x number of hexagons
- ○ 6 x number of hexagons
- ● 5 x number of hexagons + 1
- ○ 6 x number of hexagons – 4

Reflective questions for leaders to ask their teachers

When looking at and discussing the numeracy and mathematics program, you could, for example, ask the teacher:

- What differences do you notice in the range of mathematical behaviours in your classroom? In particular:
  - How many students in your classroom are able to generalise? How do you know? How are you extending them?
  - For those who are struggling to generalise, what steps are you taking to address this learning need?
- How do you model your thinking aloud to demonstrate mathematical thinking to your students? Give a recent example that seemed to be meaningful to the students.
- What strategies could you use to develop conjectures and generalisations that would enable learners to become more confident, successful mathematicians?
- What evidence of learner thinking informs your planning and your assessment of learner achievement?
- Let's consider how purposeful dialogue about generalising can facilitate collaborative learning in mathematics and numeracy.

Further resources

The big ideas in number are discussed in further detail in the following mathematics papers:

- 3.0 Conceptual understanding: Number and algebra
- 3.1 Trusting the count
- 3.2 Place value
- 3.3 Multiplicative thinking
- 3.4 Partitioning
- 3.5 Proportional reasoning.

All papers in this series are based on the work of Dianne Siemon, Professor of Mathematics Education at RMIT and a key text (Siemon et al, 2015).

http://bit.ly/BestAdviceSeries

Further reading

ACARA (2016) The tests, National Assessment Program, retrieved from https://www.nap.edu.au/naplan/the-tests This provides a link to example NAPLAN tests for Years 3, 5 , 7 and 9. Choose numeracy and look for questions requiring generalising, eg Year 3 questions 4, 30 and 33.

ACER PAT Teaching Resources Centre houses relevant concept builders for generalising, for example:

- counting collections to 100
- repeating patterns of shapes or objects
- introducing algebra.

ACER (2017) PAT Maths: Sample concept builders, Teaching Resources Centre, retrieved from https://www.acer.org/pat/pat-teaching-resources-centre/sample-content This provides a link to concept builders for patterns and algebra. Select the hyperlink 'Repeating patterns of shapes or objects' for younger students or 'Using letters to represent numbers' for older students.

Downton A, Knight R, Clarke D, & Lewis G (2006) *Mathematics assessment for learning: Rich tasks and work samples*, Fitzroy, VIC: Australian Catholic University. This practical text provides many examples of generalisations made by R–8 students.

Teaching Channel (2017) 'My favourite no: Learning from mistakes', retrieved from https://www.teachingchannel.org/videos/class-warm-up-routine (accessed March 2017)

Van De Walle JA, Karp K & Bay-Williams JM (2016) *Elementary and Middle School Mathematics: Teaching Developmentally*, Ninth global edition, UK: Pearson Education Limited. In particular, refer to chapter on 'Algebraic thinking: Generalisations, patterns and functions'.

Van de Walle JA, Karp K S, Lovin LH & Bay-Williams JM (2014) *Teaching student-centred mathematics: Developmentally appropriate instruction for grades 3–5* (2nd ed. Vol. 2), USA: Pearson Higher Ed. Chapter 15 focuses on generalisations through promoting algebraic reasoning. This book is one of a series of three. The others are Pre-K to 2 and Grades 6–8.

Victorian Department of Education and Training, Mathematics Developmental Continuum F–10 This resource provides evidence-based indicators of progress, linked to powerful teaching strategies.

Victorian Department of Education and Training, Assessment for Common Misunderstandings These tools draw on highly focussed, research-based Probe Tasks and the Probe Task Manual (RMIT), as well as a number of additional tasks and resources which have been organised to address 'common misunderstandings'.

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Siemon D, Beswick K, Brady K, Clark J, Faragher R & Warren E (2015) *Teaching Mathematics: Foundations to Middle Years*, 2nd edition, Melbourne, Oxford University Press

Victorian Department of Education and Training (2015) *True or false cards*, retrieved from http://www.education.vic.gov.au/Documents/school/teachers/teachingresources/discipline/maths/assessment/lvl6truefalse.pdf

This paper is part of the DECD Leading Learning Improvement *Best advice* series, which aims to provide leaders with the research and resource tools to lead learning improvement across learning areas within their site.

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